PERIHELION ADVANCE AND LINEARIZED FIELD EQUATION *)

K. KRAUS

Institut für Theoretische Physik der Universität

Marburg, Germany

*) Submitted to Essay Competition 1968 of the Gravity Research Foundation, New Boston (N.H.), U.S.A.

Abstract:

Contrary to a wide-spread opinion, the nonlinearities of Einstein's field equation do not significantly contribute to the perihelion advance of planetary orbits. This is demonstrated by a straightforward calculation, which yields the usual result from the linearized field equation. Roughly two-third of the effect is shown to be due to space curvature. Some general remarks on nonlinearities of classical and quantized field theories are added.

1. Introduction

Many publications on general relativity contain the statement that the perihelion advance of planetary orbits is partly caused by nonlinearities of Einstein's field equation. Usually the effect is calculated indeed from Schwarzschild's exact solution of the full nonlinear equation. The well-known result for the precession angle per revolution,

$$\alpha = \frac{6\pi m}{a(1-e^2)} = \frac{6\pi G M_\odot}{a(1-e^2)c^2} \tag{1}$$

(with a = major semi-axis, e = eccentricity, m = Schwarzschild's mass parameter, G = Newton's gravitational constant), however, is linear in G. This fact suggests the idea that perihelion advance is essentially a first order effect with respect to G or, in other words, with respect to the gravitational potential. Eq. (1) should then be expected to follow from the linearized form of Einstein's equation as well. As shown below, this is easily confirmed by a straightforward calculation.

The same conclusion has already been reached by other authors. (Compare, e.g., the footnotes on p. 296 of the first and second edition of Synge's textbook [1].) Nevertheless, in a lot of discussions the present author got the impression that most people still believe in the importance of nonlinearities for perihelion advance. Therefore the present calculation has been published for the sake of agitation, in spite of its somewhat trivial character. Perhaps just its simplicity will be especially convincing. The question of whether or not the observed perihelion advances test more than just the linear approximation of Einstein's theory is perhaps important enough to justify even a somewhat repetitious discussion.

As in the case of gravitational light deflection, the curvature of three-space produces also a significant part (two-third) of the perihelion advance. This is easily shown by a slight modification of the calculation mentioned before. Accordingly, the perihelion advance of Mercury represents up to now the most sensitive test for space curvature.

Some general considerations about nonlinearities in classical and quantum field theory are appended. As in electrodynamics, the problem of whether or not the gravitational field has to be quantized should be decidable, at least in principle, by suitably accurate observations.

2. Perihelion Advance from Weak Field Approximation

In harmonic coordinates the linearized field equation of Einstein for a static mass distribution has the solution

$$g_{ik} = \eta_{ik} - \frac{2V}{c^2}\delta_{ik} \quad . \tag{2}$$

(See, e.g., Fock [2] .) Here g_{ik} and η_{ik} are the curved and flat space-time metric, respectively, δ_{ik} is the Kronecker symbol, and V is Newton's gravitational potential. The gravitational field of the sun as given by

$$V = -\frac{GM_{\odot}}{r} = -\frac{mc^2}{r} \tag{3}$$

is more appropriately described in spherical coordinates $x^1 = r$, $x^2 = \Theta$, $x^3 = \phi$, $x^4 = ct$. Eqs. (2) and (3) then yield

$$\begin{aligned} ds^2 &= g_{ik}dx^i dx^k \\ &= \left(1 + \frac{2m}{r}\right)(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta\, d\phi^2) - \left(1 - \frac{2m}{r}\right)c^2 dt^2 \, . \end{aligned} \tag{4}$$

Planets if treated as test bodies move according to

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{k\ell}\frac{dx^k}{ds}\frac{dx^\ell}{ds} = 0 \quad . \tag{5}$$

Since g_{ik} is calculated in (4) up to terms of second and higher order in m only, such terms will consequently be omitted also in the following evaluation of Eq. (5). In this approximation, the non-vanishing Christoffel symbols calculated from (4) are

$$\Gamma^1_{11} = -\Gamma^1_{44} = -\frac{m}{r^2}\ , \quad \Gamma^1_{22} = m - r\ , \quad \Gamma^1_{33} = (m-r)\sin^2\Theta\ ,$$
$$\Gamma^2_{33} = -\sin\Theta\cos\Theta\ , \quad \Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r} - \frac{m}{r^2}\ , \quad \Gamma^3_{32} = \frac{\cos\Theta}{\sin\Theta}\ ,$$
$$\Gamma^4_{41} = \frac{g_{44}'}{2g_{44}} = \frac{m}{r^2} \qquad \left({}' \underset{\text{def.}}{=} \frac{d}{dr}\right) .$$

Inserting these Γ's, Eq. (5) for i = 2 becomes

$$\ddot{\Theta} + \left(\frac{2}{r} - \frac{2m}{r^2}\right)\dot{r}\dot{\Theta} - \sin\Theta\cos\Theta\,\dot{\phi}^2 = 0 \qquad \left({}^{\cdot} \underset{\text{def.}}{=} \frac{d}{ds}\right) ,$$

with the particular solution

$$\Theta = \frac{\pi}{2} \tag{6}$$

which just specifies the orientation of the coordinates with respect to the orbit plane. Eq. (5) for i = 3 gives, together with (6),

$$\ddot{\phi} + \left(\frac{2}{r} - \frac{2m}{r^2}\right)\dot{r}\dot{\phi} = 0 \ . \tag{7}$$

With

$$\tilde{r} = r + m \ , \tag{8}$$

multiplication of (7) by $\tilde{r}^2$ leads to

$$\frac{d}{ds}(\tilde{r}^2\dot{\phi}) = 0 \ ,$$

i.e.,

$$\left(1 + \frac{2m}{r}\right) r^2\dot{\phi} \underset{\text{def.}}{=} A = \text{const.} \tag{9}$$

Eq. (5) for i = 4 becomes

$$c\ddot{t} + \frac{b'}{b}\dot{r}c\dot{t} = 0$$

with $b = -g_{44} = 1 - 2m/r$. Multiplication by b/c gives, because of $b'\dot{r} = \dot{b}$,

$$\frac{d}{ds}(b\dot{t}) = 0 \quad ,$$

i.e.,

$$\left(1 - \frac{2m}{r}\right)\dot{t} \underset{\text{def.}}{=} \frac{B}{c} = \text{const.} \tag{10}$$

Finally, Eq. (5) for i = 1 will be replaced by the first integral of (5)

$$g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = -1 \quad ,$$

which together with (6) gives

$$\left(1 + \frac{2m}{r}\right)\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \left(1 - \frac{2m}{r}\right)c^2\dot{t}^2 = -1 \ . \tag{11}$$

For a discussion of perihelion advance it suffices to know the orbit $r = r(\phi)$. With $\dot{r} = \frac{dr}{d\phi}\dot{\phi}$ inserted into (11), and then $\dot{\phi}$ and $\dot{t}$ eliminated by (9) and (11), respectively, one has

$$\left(1 - \frac{2m}{r}\right)\left(\frac{A^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{A^2}{r^2}\right) - \left(1 + \frac{2m}{r}\right)B^2 + 1 = 0 \ .$$

Multiplication by $(1 - 2m/r)A^{-2}$ and introduction of a new radial coordinate $\tilde{r}$ according to (8) yields *)

$$\left(\frac{d}{d\phi}\frac{1}{\tilde{r}}\right)^2 + \frac{1}{\tilde{r}^2} - \frac{2m}{\tilde{r}^3} - \frac{B^2 - 1}{A^2} - \frac{2m}{A^2\tilde{r}} = 0 \ .$$

By differentiation with respect to ϕ it follows

---

*) Our radial coordinate r is the "harmonic" one [2] , whereas the radial coordinate which occurs in the usual form of Schwarzschild's solution [3] corresponds to our $\tilde{r}$. The definition of perihelion obviously does not depend on whether $\tilde{r}$ or r is used.

$$\frac{d}{d\phi}\frac{1}{\tilde{r}}\left(\frac{d^2}{d\phi^2}\frac{1}{\tilde{r}}+\frac{1}{\tilde{r}}-\frac{m}{A^2}-\frac{3m}{\tilde{r}^2}\right)=0 \ .$$

Since we are not interested in circular orbits ($\frac{d}{d\phi}\frac{1}{\tilde{r}}=0$), the final orbit equation is

$$\frac{d^2}{d\phi^2}\frac{1}{\tilde{r}}+\frac{1}{\tilde{r}}-\frac{m}{A^2}=\frac{3m}{\tilde{r}^2} \ . \tag{12}$$

This equation, however, is identical with the equation which follows without any approximations from the Schwarzschild solution (cf., e.g., [3] , Chap. 11, Eq. (11.70)). Therefore Eq. (12) yields the usual result (1) for the perihelion advance, but it has been derived here from the linearized field equation.

3. Perihelion Advance and Space Curvature

According to the discussion above, besides redshift and light deflection also the third "classical" prediction of general relativity follows from the first approximation of the theory. Among these tests the redshift is the least specific one, since it tests the equation

$$g_{44}=-1-\frac{2V}{c^2} \tag{13}$$

only, which can be justified without using the field equation (cf., e.g., [3] , Chap. 10). Therefore it is more informative to test the space part

$$g_{\alpha\beta}=\left(1-\frac{2V}{c^2}\right)\delta_{\alpha\beta} \tag{14}$$

of Eq. (2), i.e., the space curvature due to gravity.

It is well-known that both correctures (13) and (14) contribute equal amounts to the light deflection angle, i.e., space curvature produces the famous factor 2. This factor already has been confirmed by observation, but only with an error of about ten per cent [4] . Since the perihelion advance of Mercury is known more accurately, with an error of about one per cent only [4], it looks more promising to test space curvature by this effect.

For this purpose one has to estimate the contribution of (14) to the total perihelion advance. This is easily done by calculating, again to first order in m, planetary orbits in a metric g_{ik} which satisfies Eqs. (13) and (3), but without space curvature, i.e., $g_{\alpha\beta} = \delta_{\alpha\beta}$. The calculation, being a trivial modification of the one just performed, is left to the reader. Instead of Eq. (12) one derives the orbit equation

$$\frac{d^2}{d\phi^2}\frac{1}{r} + \frac{1}{r} - \frac{m}{A^2} = m\left(\left(\frac{d}{d\phi}\frac{1}{r}\right)^2 + \frac{1}{r^2}\right) , \qquad (15)$$

the right hand side describing the relativistic corrections to Newton's theory being somewhat modified.

Without these corrections the orbit becomes a Kepler ellipse

$$\frac{1}{r} = \frac{m}{A^2}\left(1 + e\cos\phi\right) = \frac{1}{a}\left(\frac{1 + e\cos\phi}{1 - e^2}\right) \qquad (16)$$

with major semi-axis a and eccentricity e. For planetary orbits e is very small such that terms of order e^2 may be neglected. Therefore $\left(\frac{d}{d\phi}\frac{1}{r}\right)^2$ may be omitted in Eq. (15). The resulting equation

$$\frac{d^2}{d\phi^2}\frac{1}{r} + \frac{1}{r} - \frac{m}{A^2} = \frac{m}{r^2} \qquad (17)$$

obviously yields one-third of the total perihelion advance as derived from (12).

Two-third of the relativistic perihelion advance in case $e^2 \ll 1$ is therefore due to space curvature. The observations of Mercury are thus, up to now, the most precise test of Eq. (14).

4. Concluding Remarks

Effects due to nonlinearities of the gravitational field equation have been not yet detected, and there is little hope to achieve this because gravitation is so extremely weak. For the same reason even the linear approximation has been tested up to now for static fields only. The search for dynamical effects like gravitational waves has not been successful so far. Imagine for the moment, however, observations of nonlinear effects to be feasible. Can one then expect them to confirm the predictions of Einstein's classical field equation? Even if one firmly believes in the correctness of this equation, the question should not a priori be answered in the affirmative.

Let us explain this by the example of the electromagnetic field interacting with the Dirac field. The corresponding classical field theory is, due to the interaction, a nonlinear one like Einstein's theory of gravitation. However, it does not correctly predict the nonlinear effects of the field since it applies to nature only after quantization. For instance, two crossed beams of light do not scatter according to the classical equations, whereas quantum electrodynamics predicts such scattering. In this case there is little doubt about which prediction will be confirmed by a suitably accurate experiment.

Presumably the nonlinearities of Einstein's theory too will lead to correct predictions only after the theory has been quantized. According to this point of view, the question of whether or not the field of gravity has to be quantized can be decided, at least in principle, by the observation of suitable nonlinear effects.

References:

[1] J.L. SYNGE, Relativity: The General Theory. North Holland Publ. Comp.; 1st Edition 1960, 2nd Edition 1964.

[2] V. FOCK, The Theory of Space Time and Gravitation. Pergamon Press 1959.

[3] L.D. LANDAU and E.M. LIFSHITZ, The Classical Theory of Fields (Course of Theoretical Physics Vol.2). Addison-Wesley Publ. Comp. and Pergamon Press 1959.

[4] Cf., e.g., R.H. DICKE in: C. DE WITT and B.S. DE WITT (Editors), Relativity, Groups and Topology (Les Houches Lectures 1963), p. 189 - 193.